ANALYTICAL WEIGHT MINIMIZATION OF TRUSSES USING CYLINDRICAL ALGEBRAIC DECOMPOSITION

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Abstract. *We present a method for the analytical evaluation of globally optimal solutions for the minimum weight design of trusses. The basis of the methodology is the Cylindrical Algebraic Decomposition (CAD) algorithm, in tandem with powerful symbolic computation for the discovery of stationary points. Certain final answers to well-known problems are produced, while future improvements in both the algorithm implementation and the computer capabilities may allow the solution of even more difficult problems. To the best of our knowledge, no similar attempt can be found in the literature.*

1 INTRODUCTION

Since the advent of inexpensive computing, structural design optimization has been a topic of huge interest among researchers. The aim is to design a structure so that it is optimal in some sense (most commonly, its weight/cost) while satisfying a number of constraints related to its safety, integrity, stiffness, or any other property. In most real-life cases, featuring multimodality and non-convex feasible regions, the use of simple gradient methods is problematic. This led to mathematical programming (MP) and optimality criteria (OC) methods, which have been used extensively in the past [1]. In MP, direct minimization is attempted e.g. using decomposition into a sequence of linear programming problems. In OC methods, assumptions on the conditions in the optimum state (e.g. "fully stressed design") provide simple recursion formulas for redesign. Nowadays, metaheuristic algorithms [2] have emerged as the best way for solving complex optimization problems. These algorithms are usually inspired by evolution, swarm intelligence or physical phenomena principles.

A different path is followed in this study, as analytical methods are employed for the discovery of globally optimum solutions. In this effort, key is the use of Cylindrical Algebraic Decomposition algorithm, proposed by Collins [3]. The algorithm, originally motivated for use in quantifier elimination but later employed in diverse fields, has been implemented in several symbolic computation programs including Mathematica (developed by Strzeboński [4]), QEPCAD [5], RedLog [6], among others. Using powerful symbolic computation, we are able to provide optimum solutions to non-trivial problems. These solutions are the final answers to specific benchmark problems and can be used for comparison purposes. To the best of our knowledge, no similar attempt can be found in the literature.

2 OBJECTIVE FUNCTION

The weight of the truss is used as objective function:

$$f(\mathbf{x}) = \sum_{i=1}^{D} \left(L_i x_i \rho_i \right), \tag{1}$$

where, D = number of design variables (groups of bars), $\mathbf{x} = \{x_1, x_2, ..., x_D\}$ = a vector containing the cross-sectional areas; L_i and ρ_i = total length and specific weight corresponding to group i. The optimization problem can be stated as the minimization of f when:

(a) $\mathbf{x}$ is subjected to the following side constraints:

$$\mathbf{x}_L \le \mathbf{x} \le \mathbf{x}_U, \tag{2}$$

where, $\mathbf{x}_L$ and $\mathbf{x}_U$ are vectors defining the minimum and maximum allowable areas, respectively (vector inequalities apply element-wise); and

(b) additional stress, buckling stress and displacement constraints are imposed, depending on the definition of the problem.

3 CYLINDRICAL ALGEBRAIC DECOMPOSITION

Given a finite set $P \subset R[x_1, x_2, ..., x_n]$ of polynomials in n variables, a P-invariant cylindrical algebraic decomposition is a special partition of $\mathbb{R}^n$ into components, called *cells*, over which each of the polynomials from P has constant sign on each cell of the decomposition [7]. The cylindrical algebraic decomposition (CAD) algorithm is an algorithmic procedure proposed by Collins [3] which constructs these decompositions; it also provides a point in each cell, called sample point, which can be used to determine the sign of the polynomials [7].

Further, given a logical combination of polynomial equations and inequalities in n real unknowns, one can use the CAD algorithm to find a cylindrical algebraic decomposition of its solution set [4]. This is applicable when the objective function and the constraints are real algebraic functions [8]. A downside of the method is its doubly exponential complexity in the number of variables [8]. This decomposition provides the feasible domain in a suitable form for exact global optimization.

4 BENCHMARK PROBLEMS

4.1 3-bar truss

The 3-bar truss is a widely used example which appears in several variations. The arrangement of the truss shown in Figure 2 is studied by Ray and Saini [10], Ray and Liew [11], Hernendez [12] and Liu et al. [13], among others. Only stress constraints are taken into account. Formally, the problem is stated as follows:

$$
\begin{aligned}
&\text{Minimize } f(x_1, x_2) = \left(2\sqrt{2}x_1 + x_2\right)L \text{ subject to} \\
&c_1(x_1, x_2) = \frac{\sqrt{2}x_1 + x_2}{\sqrt{2}x_1^2 + 2x_1x_2}P - \sigma \le 0 \\
&c_2(x_1, x_2) = \frac{x_2}{\sqrt{2}x_1^2 + 2x_1x_2}P - \sigma \le 0 \\
&c_3(x_1, x_2) = \frac{1}{x_1 + \sqrt{2}x_2}P - \sigma \le 0 \\
&\text{where } 0 < x_1 < 1;\ 0 < x_2 < 1;\ L = 100cm;\ P = 2kN;\ \sigma = 2\,kN/cm^2 .
\end{aligned}
\tag{3}
$$

Applying the CAD algorithm in the sequence $\{x_1, x_2\}$ we obtain:

$$
\begin{aligned}
&\frac{1-\sqrt{2}+\sqrt{3}}{2} < x_1 < 1 \cap \\
&\frac{\sqrt{2}(x_1-1)x_1}{2x_1-1} \le x_2 < 1
\end{aligned}
\quad .
\tag{4}
$$

The above describes precisely the coordinates of all the points $(x_1, x_2) \in \mathbb{R}^2$ for which the constraints $c_1 \le 0$, $c_2 \le 0$ and $c_3 \le 0$ hold simultaneously. The algorithm could have been applied to even more constraints (representing stress, displacement or any other kind of constraint), providing its solution set into a convenient form. Note that CAD is not related to the objective function itself.

According to Eq. (4), the infimum value of the depended variable x_2 is $\sqrt{2}(x_1-1)x_1/(2x_1-1)$. This value is optimal with respect to f because $\partial f/\partial x_2 > 0$ everywhere. Note that this holds in general, since regarding the minimum weight design of trusses, according to Eq. (1) we obtain that $\partial f/\partial x_i = L_i \rho_i > 0\ \forall i \in \{1, 2, ..., D\}$. We substitute the infimum of the depended variable x_2 in the objective function to obtain:

$$f_1(x_1) = \frac{\sqrt{2}x_1(3x_1-1)}{2x_1-1}L. \tag{5}$$

The minimum of $f_1(x_1)$ for $\left(1-\sqrt{2}+\sqrt{3}\right)/2 < x_1 < 1$ is obtained for $df_1/dx_1 = L\sqrt{2}\left(1+6(x_1-1)x_1\right)/(1-2x_1)^2 = 0$, which yields the single analytical solution $x_1^{\text{opt}} = \left(3+\sqrt{3}\right)/6$ in the relevant range. Note that $d^2 f_1/dx_1^2 = 2L\sqrt{2}/(2x_1-1)^3 > 0 \quad \forall x_1 \in \left[\left(1-\sqrt{2}+\sqrt{3}\right)/2, 1\right]$. Based on (4) and (5), it follows that $x_2^{\text{opt}} = 1/\sqrt{6}$ and $\min f_1 = \min f = L\left(\sqrt{2}+\sqrt{3/2}\right)$.

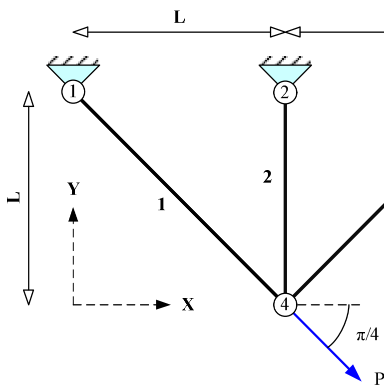

Figure 2: 3-bar truss

A graph of the constraints, the feasible domain and the contours of the objective function is shown in Figure 3. Only constraint $c_1 \le 0$ is active. The results for the 3-bar truss are summarized in Table 1 and compared to the results produced by other researchers. The ellipses ("…") at the end of a number signify that the result is not rounded, but can be evaluated to arbitrary precision.

Alternatively, we could apply the CAD algorithm in the opposite sequence $\{x_2, x_1\}$ to obtain:

$$\begin{aligned} &0 < x_2 < 1 \cap \\ &\frac{1-\sqrt{2}x_2+\sqrt{1+2x_2^{\,2}}}{2} \le x_1 < 1 \end{aligned}. \tag{6}$$

We substitute the infimum of the depended variable x_1 in the objective function:

$$f_2(x_2) = \left(\sqrt{2} - x_2 + \sqrt{2+4x_2^{\,2}}\right)L. \tag{7}$$

The minimum of $f_2(x_2)$ for $0 < x_2 < 1$ is obtained for $df_2/dx_2 = 0$ which, of course, leads to the same results. This is related to the fact that changing the order of the variables produces a different description of the same solution set. Nevertheless, problems easily solved using one variable ordering can be unsolvable with another. In [9], a number of problems is presented where one ordering leads to a cell count constant in the number of variables and another to one doubly exponential.

	Ray & Saini [10]	Ray & Liew [11]	Hernendez [12]	Liu et al. [13]	This study (analytically)	This study (numerically)
x_1	0.795	0.7886210370	0.788	0.788675134746	$\left(3+\sqrt{3}\right)/6$	0.788675…
x_2	0.395	0.4084013340	0.408	0.408248290037	$1/\sqrt{6}$	0.408248…
min f	264.3	263.8958466	263.9	263.895843376468	$L\left(\sqrt{2}+\sqrt{3/2}\right)$	263.895843…

Table 1: Results for the 3-bar truss

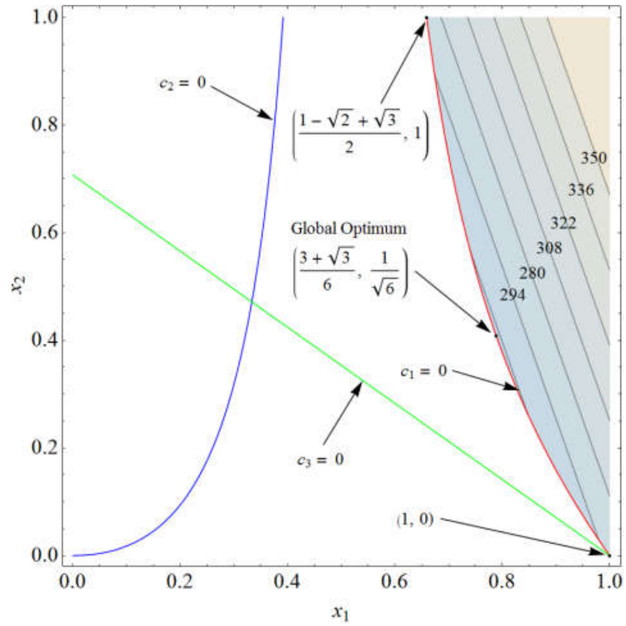

Figure 3: Feasible domain, constraints and objective function contours for the 3-bar truss.

4.2 18-bar truss

Figure 4 shows the geometry and loading of a planar truss consisting of eighteen bars and eleven nodes. The grid size is $L = 250$ in. and the point load at each free top node is $P = 20$ kips. All bars are constructed from the same material with an elastic modulus of $E = 10,000$ ksi and mass density of $\rho = 0.1$ lb/in3. The stress constraint is $\sigma_{\max} = -\sigma_{\min} = 20$ ksi for both the tension and compression members. The Euler buckling constraint is also taken into account for compression members. The buckling stress for the i th member is calculated as:

$$\sigma_i^E = -\frac{\beta E A_i}{L_i^2}\,. \tag{8}$$

where L_i, $A_i =$ length and area of the i th member, respectively, and $\beta = 4$ a constant that is determined from the geometry. The number of independent size variables is reduced to four groups as follows: (i) bars 1, 4, 8, 12 and 16; (ii) bars 2, 6, 10, 14 and 18; (iii) bars 3, 7, 11 and 15; (iv) bars 5, 9, 13 and 17. The minimum and maximum cross-sectional areas of each member are $A_{\min} = 0.10$ in^2 and $A_{\max} = 50$ in^2, respectively.

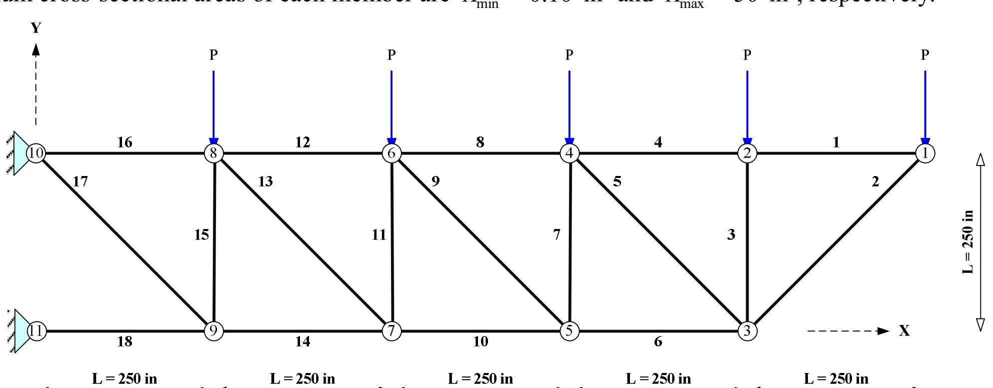

Figure 4: 18-bar truss.

This problem has previously been presented by several researchers as an example of size optimization (as in Lee & Geem [14] and Sonmez [15]), or size and shape optimization (as in Lamberti [16]). An important observation is that the truss is statically determinate. This means that the force acting on each bar is independent of the sizing configuration and can be evaluated analytically by simple equilibrium equations. The optimal solution is provided in the Appendix, both with and without member grouping. In light of this, size and shape optimization can be reduced to shape optimization only.

Optimization of the 18-bar truss becomes more challenging when considering displacement constraints instead of stress constraints. In the following analysis, we use member grouping and optimize the truss while limiting the vertical displacement of the tip of the cantilever (node 1) to 6 inches at most. Using standard analysis, the vertical displacement of node 1 is given by:

$$u_1^y = -\frac{33}{A_i} - \frac{139/2+\sqrt{2}}{A_{ii}} - \frac{6}{A_{iii}} - \frac{14\sqrt{2}}{A_{iv}}, \tag{9}$$

where, the result is given in inches, positive displacement is upwards, and A_i to A_{iv} = areas of member groups 1 to 4, given in square inches. Using standard notation, the optimization problem is formally stated as follows:

$$\begin{aligned}
&\text{Minimize } f(x_1,x_2,x_3,x_4) = 125x_1 + \left(100+25\sqrt{2}\right)x_2 + 100x_3 + 100\sqrt{2}x_4 \\
&\text{subject to} \\
&c_1(x_1,x_2,x_3,x_4) = \frac{33}{x_1} + \frac{139/2+\sqrt{2}}{x_2} + \frac{6}{x_3} + \frac{14\sqrt{2}}{x_4} \le 6 \\
&\text{where} \\
&1/10 < x_1 < 35;\ 1/10 < x_2 < 35;\ 1/10 < x_3 < 35;\ 1/10 < x_4 < 35.
\end{aligned} \tag{10}$$

Applying the CAD algorithm on the constraints of Eq. (10) in the sequence $\{x_1,x_2,x_3,x_4\}$, we obtain:

$$\begin{aligned}
&\frac{3300}{449-30\sqrt{2}} < x_1 < 50 \cap \\
&\frac{3475x_1+50\sqrt{2}x_1}{294x_1-14\sqrt{2}x_1-1650} < x_2 < 50 \cap \\
&\frac{300x_1x_2}{300x_1x_2-3475x_1-50\sqrt{2}x_1-1650x_2-14\sqrt{2}x_1x_2} < x_3 < 50 \cap \\
&\frac{28\sqrt{2}x_1x_2x_3}{12x_1x_2x_3-12x_1x_2-139x_1x_3-2\sqrt{2}x_1x_3-66x_2x_3} \le x_4 < 50
\end{aligned} \tag{11}$$

We substitute the infimum of the last variable x_4 in the objective function to obtain:

$$\begin{aligned}
&f_{123}(x_1,x_2,x_3) = \\
&25\left(\left(4+\sqrt{2}\right)x_2 + 4x_3 + x_1\left(5+\frac{224x_2x_3}{-66x_2x_3+x_1\left(12x_2(x_3-1)-\left(139+2\sqrt{2}\right)x_3\right)}\right)\right)
\end{aligned} \tag{12}$$

Solving for the stationary points of f_{123} with respect to x_3 we obtain two solutions, one of which produces invalid (negative) x_3 in the region of x_1, x_2 defined by Eq. (11). We proceed with the valid solution:

$$\begin{aligned}
&A = \sqrt{42}\sqrt{\left(19329+556\sqrt{2}\right)x_1^2 - 12\left(139+2\sqrt{2}\right)x_1(2x_1-11)x_2 + 36(11-2x_1)^2x_2^2} \\
&x_3^{\text{opt}} = \frac{132x_1x_2}{x_1\left(417+6\sqrt{2}-36x_2\right)+198x_2+A}
\end{aligned} \tag{13}$$

and substitute it in Eq. (12) to obtain:

$$\begin{aligned}
&A = \left(556+8\sqrt{2}\right)(30x_1-97)x_1x_2 - 5\left(19329+556\sqrt{2}\right)x_1^2 - 12(2x_1-11)(30x_1-29)x_2^2 \\
&B = 32\sqrt{42}x_2\sqrt{\left(19329+556\sqrt{2}\right)x_1^2 - 12\left(139+2\sqrt{2}\right)(2x_1-11)x_1x_2 + 36(11-2x_1)^2x_2^2} \\
&C = -\left(19329+556\sqrt{2}\right)x_1^2 + 12\left(139+2\sqrt{2}\right)x_1(2x_1-11)x_2 - 36(11-2x_1)^2x_2^2 \\
&f_{12}(x_1,x_2) = 25\left(\left(4+\sqrt{2}\right)x_2 + \frac{A-B}{C}x_1\right)
\end{aligned} \tag{14}$$

Figure 5 shows the contours of f_{12} at the feasible domain of x_1, x_2. Solving for the stationary points of f_{12} with respect to x_2 we obtain four solutions, etc. Finally, after analyzing all branches of the solution tree, rejecting the invalid stationary points and sorting the valid ones by their objective function value, we obtain the global optimum, which is presented in Table 2.

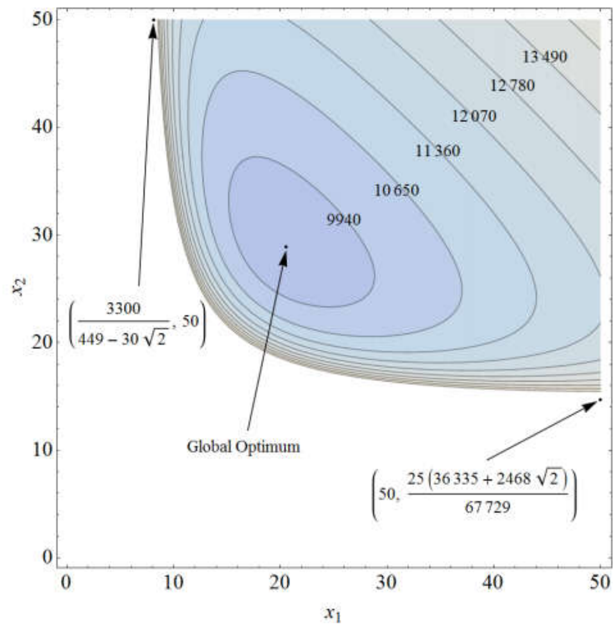

Figure 5: Feasible domain and contours of objective function f_{12} for the size optimization of 18-bar truss.

4 CONCLUSIONS

In this study, analytical methods are employed to discover exact, globally optimal solutions for the minimum weight design of trusses. The basis of the methodology is the Cylindrical Algebraic Decomposition algorithm, proposed by Collins [2]. Exact results to certain well-known problems are produced. The algorithm is expensive in the number of variables; however, the topic remains active among researchers in the symbolic computation field, with many improvements since the original version [6], while both the software implementations and the computer capabilities constantly improve. Facing even more difficult problems in the future seems feasible.

5 APPENDIX: OPTIMUM DESIGN OF 18-BAR TRUSS

The truss is statically determinate. The optimum area of each bar is easy to evaluate independently based on the following rule:

$$A_i^{\text{opt}} = \begin{cases} \max\left(\dfrac{N_i}{\sigma_{\max}}, A_{\min}\right), N_i \geq 0 \\ \max\left(\dfrac{N_i}{\sigma_{\min}}, \sqrt{-\dfrac{N_i}{\beta E} L_i}, A_{\min}\right), N_i < 0 \end{cases}, \tag{15}$$

where, $N_i =$ the force acting on the i^{th} bar, determined by equilibrium. Using a smaller area for any member produces stress violation; using a larger area is suboptimal. This methodology is applicable to all statically determinate trusses with stress constraints. When optimizing these trusses with evolutionary algorithms, size and shape optimization can be reduced to shape optimization only.

Regarding the 18-bar truss, described previously, the optimum areas as well as the minimum total weight are summarized in Table 3. If member grouping is removed, the results of Table 4 are obtained.

	This study (analytically)	This study (numerically)
A_i [in^2]	$A=\left(-6193440+1469820\sqrt{2}+345840\sqrt{21}-728640\sqrt{42}\right)^2$ $B=-4\left(563040-133620\sqrt{2}-31440\sqrt{21}+66240\sqrt{42}\right)$ $C=498487\sqrt{2}+445500\sqrt{21}-4276074\sqrt{42}-30468933$ $D=141873181216-29997745947\sqrt{2}$ $E=21784547520\sqrt{42}-9212274180\sqrt{21}$ $F=-44\sqrt{14(D+E)}$ $G=6193440-1469820\sqrt{2}-345840\sqrt{21}+728640\sqrt{42}$ $H=2\left(563040-133620\sqrt{2}-31440\sqrt{21}+66240\sqrt{42}\right)$ $A_i=\dfrac{G+\sqrt{A+B\times(C+F)}}{H}$	20.518856…
A_{ii} [in^2]	$\dfrac{\left(973+14\sqrt{2}+2\sqrt{131376-31178\sqrt{2}-7336\sqrt{21}+15456\sqrt{42}}\right)A_i}{42(2A_i-11)}$	28.905476…
A_{iii} [in^2]	$A=\left(19329+556\sqrt{2}\right)A_i+36(11-2A_i)^2\,A_{ii}^{\,2}$ $B=-12\left(139+2\sqrt{2}\right)(2A_i-11)A_iA_{ii}$ $C=\sqrt{A+B}$ $A_{iii}=\dfrac{132A_iA_{ii}}{A_i\left(417+6\sqrt{2}-36A_{ii}\right)+198A_{ii}+\sqrt{42}C}$	9.781980…
A_{iv} [in^2]	$\dfrac{28\sqrt{2}A_iA_{ii}A_{iii}}{12A_iA_{ii}A_{iii}-12A_iA_{ii}-139A_iA_{iii}-2\sqrt{2}A_iA_{iii}-66A_{ii}A_{iii}}$	14.942222…
min f [lb]	$125A_i+\left(100+25\sqrt{2}\right)A_{ii}+100A_{iii}+100\sqrt{2}A_{iv}$	9568.715174…

Table 2: Optimum results for the 18-bar truss with displacement constraint at the tip.

	Lee & Geem [14]	Sonmez [15]	This study (analytically)	This study (numerically)
A_i [in^2]	9.980	10.000	$10P/\sigma_{\max}$	10.
A_{ii} [in^2]	21.630	21.651	$\sqrt{15P/(\beta\,E)}L$	21.650635…
A_{iii} [in^2]	12.490	12.500	$\sqrt{5P/(\beta\,E)}L$	12.5
A_{iv} [in^2]	7.057	7.071	$5\sqrt{2}P/\sigma_{\max}$	7.071067…
min f [lb]	6421.880	6430.529	$\left(\dfrac{90P}{\sigma_{\max}}+\sqrt{5}\left(4+4\sqrt{3}+\sqrt{6}\right)L\sqrt{\dfrac{P}{E\beta}}\right)L\rho$	6430.529054…

Table 3: Optimum results for the 18-bar truss with stress constraints (with member grouping)

Member	Length (analytically)	Acting force (analytically)	Acting force (numerically) [kips]	Optimum area (analytically, based on (15))	Optimum area (numerically) [in²]
1	L	P	20.	$P/\sigma_{\max}$	1.
2	$\sqrt{2}L$	$-\sqrt{2}P$	-28.284271...	$\sqrt{\sqrt{2}P/(\beta\ E)}\sqrt{2}L$	9.401508…
3	L	$-P$	-20.	$\sqrt{P/(\beta\ E)}L$	5.590170…
4	L	P	20.	$P/\sigma_{\max}$	1.
5	$\sqrt{2}L$	$2\sqrt{2}P$	56.568542...	$2\sqrt{2}P/\sigma_{\max}$	2.828427…
6	L	$-3P$	-60.	$\sqrt{3P/(\beta\ E)}L$	9.682458…
7	L	$-3P$	-60.	$\sqrt{3P/(\beta\ E)}L$	9.682458…
8	L	$3P$	60.	$3P/\sigma_{\max}$	3.
9	$\sqrt{2}L$	$3\sqrt{2}P$	84.852813...	$3\sqrt{2}P/\sigma_{\max}$	4.242641…
10	L	$-6P$	-120.	$\sqrt{6P/(\beta\ E)}L$	13.693064…
11	L	$-4P$	-80.	$\sqrt{4P/(\beta\ E)}L$	11.180340…
12	L	$6P$	120.	$6P/\sigma_{\max}$	6.
13	$\sqrt{2}L$	$4\sqrt{2}P$	113.137084...	$4\sqrt{2}P/\sigma_{\max}$	5.656854…
14	L	$-10P$	-200.	$\sqrt{10P/(\beta\ E)}L$	17.677670…
15	L	$-5P$	-100.	$\sqrt{5P/(\beta\ E)}L$	12.5
16	L	$10P$	200.	$10P/\sigma_{\max}$	10.
17	$\sqrt{2}L$	$5\sqrt{2}P$	141.421356...	$5\sqrt{2}P/\sigma_{\max}$	7.071068…
18	L	$-15P$	-300.	$\sqrt{15P/(\beta\ E)}L$	21.650635…
min f (analytically)					min f (numerically) [lb]
$\left(\frac{49P}{\sigma_{\max}}+\left(3+2\sqrt[4]{2}+2\sqrt{3}+\sqrt{5}+\sqrt{6}+\sqrt{10}+\sqrt{15}\right)L\sqrt{\frac{P}{E\beta}}\right)L\rho$					4098.813372…

Table 4: Optimum results for the 18-bar truss with stress constraints (without member grouping).

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