9th HSTAM International Congress on Mechanics
Limassol, Cyprus, 12 – 14 July, 2010

# PARAMETERS OF BOUC-WEN HYSTERETIC MODEL REVISITED

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**Keywords:** Bouc-Wen, hysteresis, parameter analysis.

**Abstract.** *In this work, the parameters of Bouc-Wen hysteretic model are examined in detail. Their effect on the overall response is clarified and discussed. The analysis is based on both mathematical and physical requirements, as well as on the analytical relations for the response and the dissipated energy of the model that were derived recently.*

## 1 INTRODUCTION

The Bouc-Wen hysteretic model is well-known and very popular because of its versatility and simplicity. It is a very concise model governed by a single differential equation that can be easily applied in several hysteretic phenomena in the fields of magnetism, electricity, materials and elasto-plasticity of solids. It was first introduced by Bouc[1] in 1967, but it was Wen[2] who extended the model by producing a variety of hysteretic patterns.

Although the model has been routinely used for several decades, there is still a certain ambiguity concerning its controlling parameters. In this work, the effect of these parameters is analyzed and discussed in detail.

## 2 MODEL FORMULATION

The restoring force $F(t)$ of a single-degree-of-freedom system can be expressed as:

$$F = a\frac{F_y}{u_y}u + (1-a)F_y z \tag{1}$$

where, $u(t)$ is the displacement, $F_y$ the yield force, $u_y$ the yield displacement, $a$ the ratio of post-yield to pre-yield (elastic) stiffness and $z(t)$ a dimensionless hysteretic parameter that obeys a single non-linear differential equation with zero initial condition:

$$\dot{z} = \frac{A - |z|^n\left(\beta + \mathrm{sgn}(\dot{u}\,z)\gamma\right)}{u_y}\dot{u} \tag{2}$$

where, $A$, $\beta$, $\gamma$, $n$ are dimensionless quantities controlling the behavior of the model, $sgn(\cdot)$ is the signum function and the overdot denotes the derivative with respect to time. The reader is cautioned, as in the work of some other researchers the symbols for parameters $\beta$ and $\gamma$ are exchanged due to different mathematical formulation.

It follows from Eq. (1) that the restoring force $F(t)$ can be analyzed into an elastic and a hysteretic part as follows:

$$F^{el} = a\frac{F_y}{u_y}u \tag{3}$$

$$F^{h} = (1-a)F_y z \tag{4}$$

Thus, the model can be visualized as two springs connected in parallel (Figure 1) where, $k_i=F_y/u_y$ and $k_f=a\,k_i$ are the initial and post-yielding stiffness of the system.

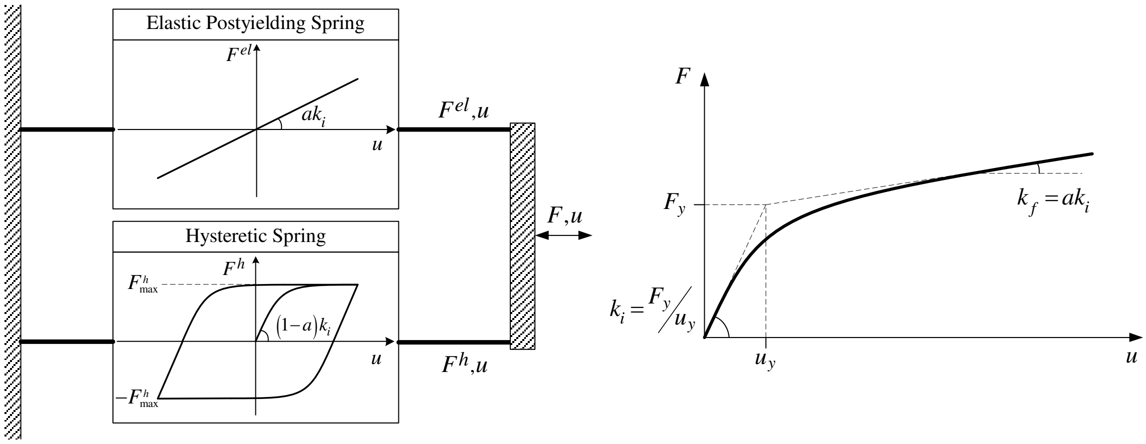


Figure 1. Bouc-Wen model.

## 3 MATHEMATICAL CONSISTENCY

The parameters of Bouc-Wen model are functionally redundant; there exists a multiplicity of parameter vectors that produce an identical response for a given excitation[3]. Removing this redundancy can be achieved by fixing one of the parameters to a specific value[3]. For many reasons, the best choice is to fix parameter $A$ to unity. For instance, when $A$=1 the physical meaning of the initial stiffness $k_i=F_y/u_y$ is restored. To show this, based on Eq. (2) and observing that the initial condition for parameter $z$ is zero, we derive that:

$$\left.\frac{dz}{du}\right|_{t=0} = \frac{A}{u_y} \tag{5}$$

Also, based on Eq. (1) it follows that:

$$\frac{dF}{du} = a\frac{F_y}{u_y} + (1-a)F_y\frac{dz}{du} \tag{6}$$

Combining these relations, we derive that the actual (i.e. the observed) initial stiffness $k^*_i$ exhibited by the system is:

$$k_i^* = \left.\frac{dF}{du}\right|_{t=0} = a\frac{F_y}{u_y} + (1-a)\frac{F_y}{u_y}A \tag{7}$$

Setting the *actual* initial stiffness $k^*_i$ equal to the *assumed* initial stiffness $k_i=F_y/u_y$, and given that $a\neq 1$ (when $a$=1, the Bouc-Wen model degenerates to a linear system), we derive a sufficient and necessary condition:

$$A = 1 \tag{8}$$

Henceforth, Eq. (8) is assumed to hold. Considering system identification, the issue of parameter redundancy is particularly critical[4] and must be treated. Thus, it is our belief that parameter $A$ should be fixed outright, rather than elaborated in mathematical analyses or allowed to mitigate in modern Bouc-Wen type models. It is noted that some researchers have presented normalized versions of the model which treat the parameter redundancy. However, the new parameters that are introduced may not have clear physical representation, e.g. [5], [6]. In the Author's point of view, such an approach is not preferable to imposing $A$=1 in the original model.

## 4 MODEL PARAMETERS

### 4.1 Parameters $\beta$ and $\gamma$

Parameters $\beta$ and $\gamma$ control the shape and size of the hysteretic loop, as demonstrated by Wen[3]. However, these parameters do not have physical interpretation and affect the whole response in an indirect and unclear manner. In illustration, Figure 2 shows two examples taken from the original paper[3], along with the corresponding bilinear models:

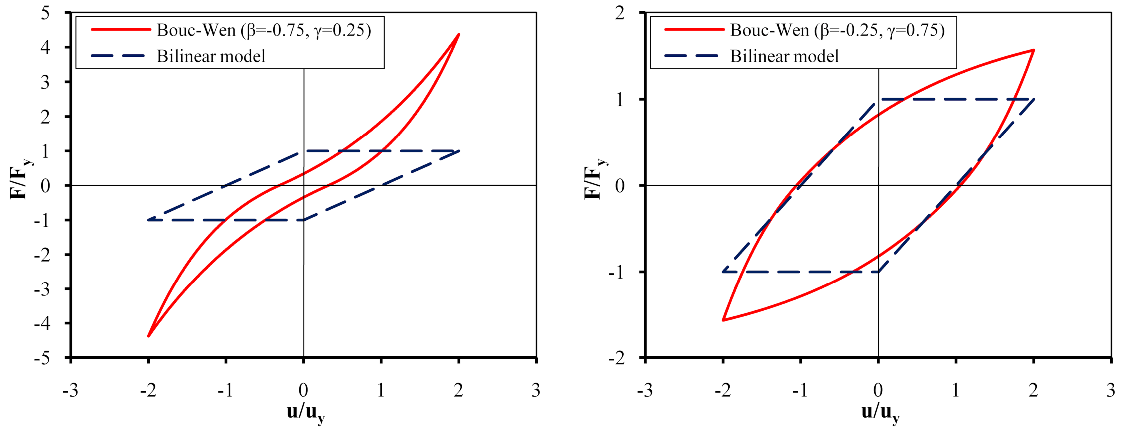


Figure 2. Hysteretic loops with $a$=0, $n$=1, $u_{max}/u_y$=2 and
(a) strain hardening ($\beta$=-0.75, $\gamma$=0.25), (b) strain softening ($\beta$=-0.25, $\gamma$=0.75).

Whether the system exhibits strain-hardening or strain-softening, it is obvious that the responses of the Bouc-Wen model are radically different from those of the corresponding bilinear model. Further, the effect of the rest of the parameters ($F_y$, $u_y$, $a$, $n$) is uncertain. In this context, descriptions such as "yield force", "yield displacement", "ratio of post-elastic to elastic stiffness" etc are pointless, as the parameters do not bear any clear physical meaning. The cases shown in Figure 2 may present some interest in terms of mathematics. In engineering, however, having physically defined model parameters is always a strong advantage, e.g. for identification purposes.

Early studies by Constantinou and Adnane[7] suggested imposing a certain constraint, viz. $A/(\beta+\gamma)$=1, to reduce the model to a strain-softening formulation with well-defined properties. Given Eq. (8), this results to:

$$\beta + \gamma = 1 \tag{9}$$

The same constraint is adopted henceforth. To show how the two adopted constraints, i.e. Eqs. (8) and (9), affect the response, we first derive the necessary condition for strain-softening behavior. Under monotonic loading, we assume that the hysteretic parameter $z$ exhibits some maximum value. This maximum value is derived by setting $dz/dt$=0 in Eq. (2), so it follows that:

$$z_{\max} = \left( \frac{A}{\beta + \gamma} \right)^{1/n} \tag{10}$$

Given that $n$>0, Eq. (10) is valid only when $A/(\beta+\gamma)$>0 or (taking into account Eq. (8)):

$$\beta + \gamma > 0 \tag{11}$$

Eq. (11) is a sufficient and necessary condition for strain-softening behavior. When $\beta+\gamma$<0, $z$ exhibits no extrema during monotonic loading (i.e. it increases or decreases continuously).

Further, we seek to restore the physical meaning of the rest of the parameters. For instance, we consider a yielding Bouc-Wen system under monotonic loading with no post-elastic stiffness ($a$=0). In this case, the actual yield force exhibited by the system is evaluated based on Eq. (1) as:

$$F_y^* = F_y z_{\max} \tag{12}$$

From Eq. (12), it is deducted that the *actual* yield force $F^*_y$ is equal to the *assumed* yield force $F_y$ only when $z_{max}$=1, which (based on Eq. (10)) leads to the second adopted constraint, i.e. Eq. (9).

Summarizing, Eqs. (8) and (9) reduce the Bouc-Wen model to a strain-softening formulation with well-defined mechanical properties. The former constraint is imposed for reasons of mathematical consistency, while the latter for reasons of physical consistency. When adopting these constraints, the dimensionless hysteretic parameter $z$ takes values in the range [-1,1]. When $z$=1, full yield has occurred due to loading in the positive direction; when $z$=-1 full yield has occurred due to loading in the negative direction. Intermediate values signify intermediate states of loading or unloading.

It is our belief that the use of such a crystal-clear, physically-sound model is preferable, when compared with a core model that is more versatile on the expense of ambiguous parameters. This is especially true since there are other, more efficient methods of introducing strain-hardening. One example is the introduction of a dedicated gap-closing spring, as proposed by Sivaselvan and Reinhorn[8]. Another is an extended version of the Bouc-Wen model capable of producing response curves with inflection points[9].

In another context, thermodynamic admissibility issues impose the following inequality[10]:

$$\gamma \geq \beta \tag{13}$$

It is noted that the symbols for parameters $\beta$ and $\gamma$ are exchanged with respect to [10], due to different mathematical formulation.

Eq. (13) results in bulky, ellipsoid hysteretic loops. In case $\beta>\gamma$, the hysteretic loops take the shape of an “S”. In the special case of $\beta=\gamma=1/2$, the unloading branches are straight lines with stiffness equal to $F_y/u_y$. The aforementioned three cases are depicted in Figure 3, along with the response of the corresponding bilinear model. It is noted that in all cases, the response of the Bouc-Wen model during loading in either direction tends asymptotically to the one of the bilinear model. This is due to the adopted constraints, i.e. Eqs. (8) and (9).

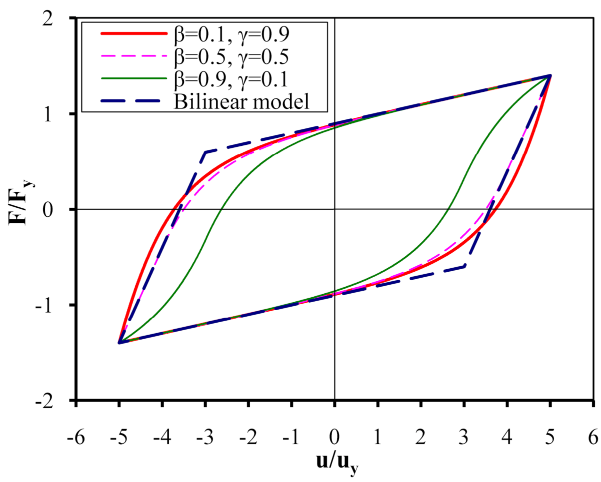


Figure 3. Hysteretic loops ($a$=0.10, $n$=1, $u_{max}/u_y$=5).

### 4.2 Parameter *n*

The exponential parameter $n$ governs the abruptness of the transition between elastic and post-elastic branch. Figure 4 shows the hysteretic loops corresponding to various values of $n$. For large values, the response approaches that of the bilinear model.

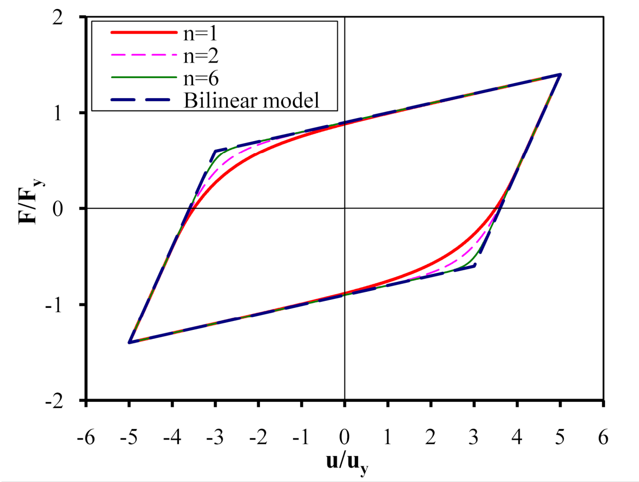


Figure 4. Hysteretic loops for various values of $n$ ($\beta$=$\gamma$=0.5, $a$=0.10, $u_{max}/u_y$=5).

### 4.3 Parameter *a*

Parameter $a$ is the ratio of post-elastic to elastic stiffness. Figure 5 shows the responses of two Bouc-Wen models featuring $a$=0 and $a$=0.10, as well as the responses of the corresponding bilinear models. When $a$=0, the whole Bouc-Wen model is expressed by the hysteretic spring only (Figure 1).

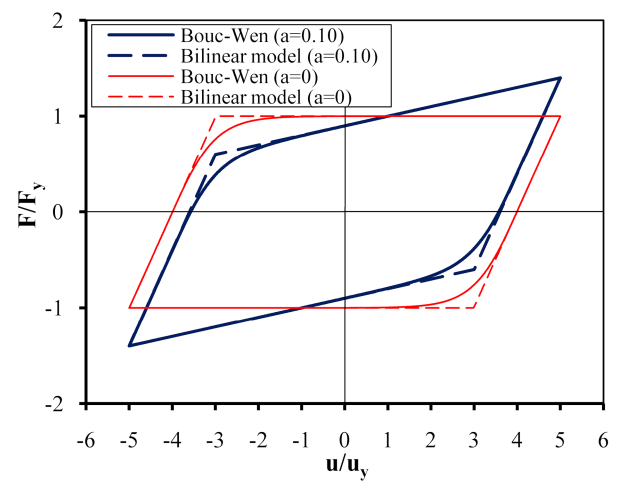


Figure 5. Hysteretic loops for $a$=0 and $a$=0.10 ($\beta$=$\gamma$=0.5, $n$=2, $u_{max}/u_y$=5).

### 4.4 Parameters $F_y$ and $u_y$

Parameters $F_y$ and $u_y$ are entitled "yield force" and "yield displacement", respectively, although this is not entirely accurate. The reason is that, in general, the "yield point" ($u_y$, $F_y$) does *not* belong to the response of the Bouc-Wen model under monotonic loading, even when $A$=$\beta$+$\gamma$=1; rather, it is the yield point of the corresponding bilinear model. Figure 6 shows the response under monotonic loading in case $n$=1 and $n\rightarrow\infty$; the latter coincides with the bilinear response.

The aforementioned point has an important implication. The coordinates of the (sometimes) well-defined yield point that is observed experimentally in an actual system cannot be used to identify the optimum yield parameters $F_y$ and $u_y$. On the contrary, special identification procedures are required to estimate the whole set of unknown parameters based on experimental data, e.g. [11].

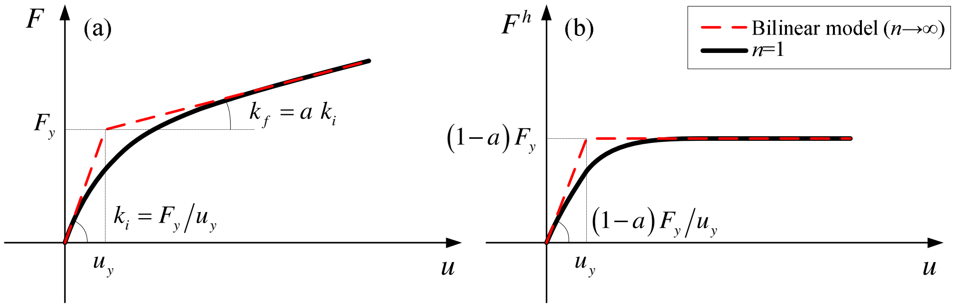


Figure 6. Response under monotonic loading (a) total (b) hysteretic only.

# 5 RESPONSE AND DISSIPATED ENERGY

## 5.1 Response

Recently, analytical relations for the response and dissipated energy were derived[12]. In illustration, the response of Bouc-Wen model can be divided into four segments depending on the sign of *du/dt* and *z* (Figure 7).

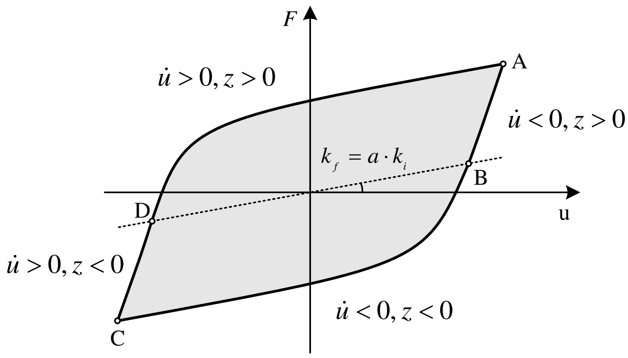


Figure 7. Response of Bouc-Wen model under cyclic excitation.

The hysteretic parameter $z$ was associated with the displacement $u$ in terms of Gauss' hypergeometric function ${}_2F_1(\cdot)$, as follows:

$$\frac{u-u_0}{u_y} = z\,{}_2F_1\left(1,\frac{1}{n},1+\frac{1}{n};q|z|^n\right)\Bigg|_{z_0}^{z} \tag{14}$$

where, $q=\beta+sgn(du/dt\cdot z)\gamma$ and $u_0$, $z_0$ are the initial values of the displacement and hysteretic parameter, respectively. Eq. (14) holds for any transition in which $q$ remains constant.

Eq. (14) explicitly provides the displacement $u$ (i.e. the input of the hysteretic operator) in terms of $z$ (the output). This is very convenient in certain cases, e.g. during the formulation of a modified Bouc-Wen model that is compatible with Drucker's and Il'iushin's postulates of plasticity[13]. Solving Eq. (14) for $z$ does not seem to be possible for an arbitrary value of $n$. For $n$=1, however, the following relation was derived[12]:

$$z = \frac{sgn(z)+\left(q\,z_0 - sgn(z)\right)e^{-\frac{sgn(z)q(u-u_0)}{u_y}}}{q} \tag{15}$$

Also, for $n$=2, one obtains[12]:

$$z = \frac{tanh\left(\sqrt{q}\left(u-u_0\right)/u_y + arctanh\left(\sqrt{q}z_0\right)\right)}{\sqrt{q}} \tag{16}$$

where, $tanh(\cdot)$, $arctanh(\cdot)$ are the normal and inverse hyperbolic tangent, respectively. In Eq. (16), the denominator may be complex but the result is real. Special attention must be paid with respect to the values of signum and the domain of hysteretic parameter $z$ per segment (Table 1).

The lack of a generic inverse relation $z=f(u)$ is not so important, because the numerical evaluation of $z$ can be performed very efficiently based on Eq. (14) using bisection-type methods[12]. In particular, the Van Wijngaarden – Dekker – Brent method[14] exhibits excellent performance. It is important to note that, considering the hypergeometric function ${}_2F_1(a,b,c;w)$, point $w(1,0)$ is singular in the complex plane and the limit needs to be evaluated as $w \to 1^-$. Proper evaluation techniques are given in [12].

| segment | q | sgn(z) | domain |
|---|---|---|---|
| AB | $\beta$-$\gamma$ | 1 | $z_0 \in [0,1],\ z \in [0, z_0]$ |
| BC | 1 | -1 | $z_0 \in (-1,0],\ z \in (-1, z_0]$ |
| CD | $\beta$-$\gamma$ | -1 | $z_0 \in [-1,0],\ z \in [z_0, 0]$ |
| DA | 1 | 1 | $z_0 \in [0,1),\ z \in [z_0, 1)$ |

Table 1: Signum value and domain of $z$ per segment

## 5.2 Dissipated energy

The dissipated energy is expressed by the area enclosed by the hysteretic loops. Generic analytical expressions of the dissipated energy were derived with respect to the steady-state response under symmetric wave T-periodic input[12]. In particular, the dissipated energy $E$ during a complete symmetric cycle is given as:

$$E = 2F_{\max}^{h} u_y \left(2\frac{u_{\max}}{u_y} - k_{CD}^{*} - k_{DA}^{*}\right) \tag{17}$$

where, $u_{max}$ is the maximum observed displacement and $F^h_{max}=(1\text{-}a)F_y$ the maximum force of the hysteretic spring. Coefficients $k^*_{CD}$ and $k^*_{DA}$ are given by the following relations:

$$k_{CD}^{*} = z_A\ {}_2F_1\left(1,\frac{1}{n},1+\frac{1}{n};(\beta-\gamma)z_A{}^n\right) + \frac{1}{2}z_A{}^2\ {}_2F_1\left(1,\frac{2}{n},1+\frac{2}{n};(\beta-\gamma)z_A{}^n\right) \tag{18}$$

$$k_{DA}^{*} = z_A\ {}_2F_1\left(1,\frac{1}{n},1+\frac{1}{n};z_A{}^n\right) - \frac{1}{2}z_A{}^2\ {}_2F_1\left(1,\frac{2}{n},1+\frac{2}{n};z_A{}^n\right) \tag{19}$$

where, $z_A$ is the maximum observed value of the hysteretic parameter (corresponding to point A of Figure 7). In illustration, Figure 8 presents the dissipated energy as a function of the displacement amplitude for several values of the exponential parameter $n$. Both axes are normalized, while parameter $\gamma$ is taken equal to 0.9. It is observed that, as the displacement amplitude increases, all curves become straight and parallel lines with a common slope equal to four. This is expected because, for a fully yielding system, an increase in displacement amplitude $\Delta u_{max}$ would result in an increase $\Delta E=4F^h_{max}\Delta u_{max}$ of the area enclosed by the hysteretic loop. Further, as parameter $n$ is increased, the response of the system approaches that of a bilinear model and thus the dissipated energy diminishes for $u_{max}/u_y<1$.

When the system yields fully ($z_A=1$), which is the case of most interest, the coefficients $k^*_{CD}$ and $k^*_{DA}$ can be provided with sufficient accuracy by the following simple relations:

$$k_{CD}^{*} \cong \frac{0.003\ln(n) - 1.784\ln(\gamma) - 1.238}{1 + 0.89n + 0.592\gamma} + 1.5,\ n \in [0.5,12],\ \gamma \in [0.5,1.0] \tag{20}$$

$$k_{DA}^{*} \cong \frac{126.57 + 87.66n + 35.96n^2}{1.0 + 177.37n + 71.83n^2},\ n \in [0.5,12] \tag{21}$$

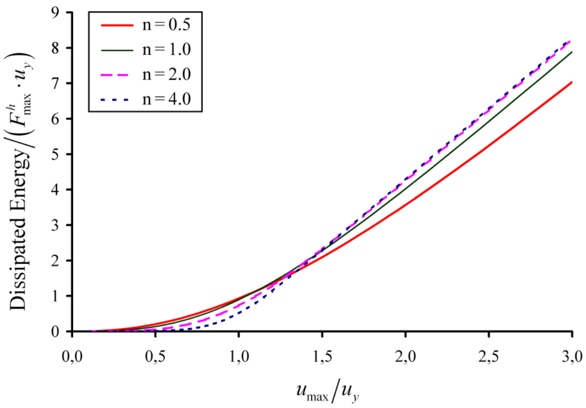


Figure 8. Dissipated energy as a function of displacement amplitude ($\beta$=0.1, $\gamma$=0.9).

## 6. CONCLUSIONS

In this work, the effect of the Bouc-Wen model parameters was clarified. In addition, certain issues concerning the mathematical and physical consistency of the model have been analyzed and discussed. Proper and effective solutions to these issues have been presented, which reflect the Author's point of view.

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